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# Reachability of standard and fractional continuous-time systems with constant inputs

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The reachability of standard and fractional-order continuous-time systems with constant inputs is addressed. Positive and non-positive continuous-time linear systems are considered. Necessary and sufficient conditions for the existence of such constant inputs that steers the system from zero initial conditions to the given final state in desired time are derived and proved. As an example of such systems the electrical circuits with DC voltage sources are presented.

**Key words:** constant input, reachability, fractional system, positive system.

## 1. Introduction

Reachability is one of the most important structural property of dynamical systems. It appears in many different areas, such as computational models, cellular automata, Petri nets, decision procedures, game theory, etc. The problem of reachability of standard ordinary-order systems has been addressed in [2, 8, 17]. Reachability of linear hybrid systems described by the general model and their piecewise constant control have been considered in [15].

Positive systems is a wide class of systems in which inputs, outputs and state variables take only nonnegative values. Positive behavior occurs in engineering, economics, social sciences, biology, medicine, etc. An overview of state of the art in positive systems theory is given in the monographs [5, 10]. Reachability of positive continuous-time and discrete-time systems has been addressed in [4, 10, 24]. The problem of reachability of electrical circuits has been solved in [11, 16].

Many physical phenomena have "fractional" nature, i.e. fractional-order differential or difference equations provides more accurate descriptions than ordinary calculus do [1, 3, 7, 18]. Fundamentals of fractional calculus are given in monographs [13, 16, 19, 20]. Reachability of fractional systems and electrical circuits have been considered in [12, 14, 21, 22, 23].

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In this paper the problem of existence of constant inputs vector that steers the system from zero initial conditions to the given final state in desired time  $t_f$  will be considered. The following classes of systems will be addressed: standard (non-positive) and positive ordinary-order and fractional-order continuous-time systems.

To the best knowledge of the author reachability of fractional and positive fractional continuous-time systems when the input is a constant function has not been considered yet.

The structure of the paper is following. In section 2 the reachability of standard continuous-time systems with constant inputs is considered. Necessary and sufficient conditions for the existence of constant input that steers the systems from zero initial conditions to the given final states are given. Similar problem will be solved for fractional systems in section 3. In section 4 the reachability of positive standard and positive fractional-order systems with constant inputs is adressed. Concluding remarks are given in section 5. The considerations are illustrated by examples of electrical circuits with DC voltage sources.

The following notation will be used.  $\mathbb{R}$  is the set of real numbers,  $\mathbb{R}^{n \times m}$  – the set of  $n \times m$  matrices with real entries.  $\mathbb{R}_+^{n \times m}$  – the set of  $n \times m$  matrices with real nonnegative entries and  $\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}$ . The set of  $n \times n$  Metzler matrices (real matrices with nonnegative off-diagonal entries) will be denoted by  $M_n$  and the identity matrix of size  $n$  by  $\mathbb{I}_n$ .

## 2. Reachability of standard continuous-time linear systems with constant inputs

Let us consider the standard continuous-time system described by the state equation [8, 17]

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $u(t) \in \mathbb{R}^m$  is the input vector of the system for  $t \geq 0$  and matrices  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ .

The solution to the state equation (1) with initial condition  $x_0 = x(0) \in \mathbb{R}^n$  and input vector  $u(t) \in \mathbb{R}^m$  for  $t \geq 0$  is given by [8, 17]

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = e^{At}x_0 + \int_0^t e^{A\tau}Bu(t-\tau)d\tau. \quad (2)$$

Now, we will consider the continuous-time system (1) with constant inputs vector  $U$ , i.e.

$$u(t) = U \in \mathbb{R}^m \quad \text{for } t \geq 0. \quad (3)$$

**Definition 1** *The standard continuous-time linear system (1) is called reachable for constant inputs in time  $t_f > 0$  if there exists such constant input vector (3), that steers the system from zero initial conditions  $x_0 = x(0) = 0$  to arbitrary final state  $x_f = x(t_f) \in \mathbb{R}^n$ .*

Let us assume that  $\det A \neq 0$ . Then the following theorem holds.

**Theorem 1** *The system (1) is reachable for constant input in time  $t_f > 0$  if and only if the matrix*

$$R_c = A^{-1} (e^{At_f} - \mathbb{I}_n) B \in \mathbb{R}^{n \times m} \quad (4)$$

has  $n$  linearly independent columns, i.e.

$$\text{rank} R_c = n. \quad (5)$$

**Proof** Substituting (3) into (2) for zero initial conditions and  $t = t_f > 0$  we obtain

$$x(t_f) = x_f = \int_0^{t_f} e^{A\tau} B U d\tau = \int_0^{t_f} e^{A\tau} d\tau B U = A^{-1} (e^{At_f} - \mathbb{I}_n) B U, \quad (6)$$

since [9]

$$\int_0^{t_f} e^{A\tau} d\tau = A^{-1} (e^{At_f} - \mathbb{I}_n).$$

The equation (6) has a solution  $U$  for arbitrary final state  $x_f \in \mathbb{R}^n$  if and only if the matrix  $R_c$  satisfies the condition (5).  $\square$

The constant input that steers the system (1) from zero initial state to the final state  $x_f$  in time  $t \in [0, t_f]$  is given by the formula

$$U = R_c^+ x_f, \quad (7)$$

where  $R_c^+ \in \mathbb{R}^{m \times n}$  is the right pseudoinverse of the rectangular matrix  $R_c$  given by one of the following formulae [6, 9]

$$R_c^+ = R_c^T [R_c R_c^T]^{-1} + (\mathbb{I}_m - R_c^T [R_c R_c^T]^{-1} R_c) K_1 \quad \text{for arbitrary } K_1 \in \mathbb{R}^{m \times n}; \quad (8a)$$

$$R_c^+ = K_2 [R_c K_2]^{-1} \quad \text{for arbitrary } K_2 \in \mathbb{R}^{m \times n}, \det [R_c K_2] \neq 0. \quad (8b)$$

**Example 1** Consider the electrical circuit shown in Fig. 1 with given resistances  $R_1 = 0.1\Omega$ ,  $R_2 = 1\Omega$ ,  $R_3 = 2\Omega$ , inductance  $L = 0.5\text{H}$  and capacitance  $C = 0.1\text{F}$ .

Using Kirchhoff's laws we may formulate the state equations (1) of the circuit shown in Fig. 1

$$\frac{d}{dt} \begin{bmatrix} i_L(t) \\ u_C(t) \end{bmatrix} = A \begin{bmatrix} i_L(t) \\ u_C(t) \end{bmatrix} + B \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}, \quad (9a)$$

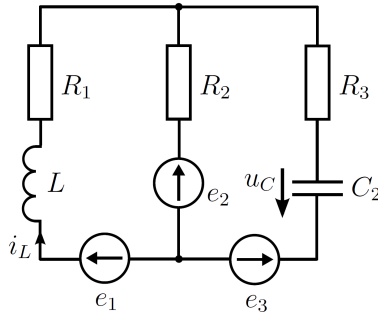


Figure 1. Electrical circuit of Example 1.

where

$$A = \begin{bmatrix} -\frac{R_1}{L} - \frac{R_2 R_3}{L(R_2 + R_3)} & \frac{R_2}{L(R_2 + R_3)} \\ -\frac{R_2}{C(R_2 + R_3)} & -\frac{1}{C(R_2 + R_3)} \end{bmatrix} = \begin{bmatrix} -1.53 & 0.67 \\ -3.33 & -3.33 \end{bmatrix}, \quad (9b)$$

$$B = \begin{bmatrix} \frac{1}{L} & -\frac{R_3}{L(R_2 + R_3)} & \frac{R_2}{L(R_2 + R_3)} \\ 0 & -\frac{1}{C(R_2 + R_3)} & \frac{1}{C(R_2 + R_3)} \end{bmatrix} = \begin{bmatrix} 2 & -1.33 & -0.67 \\ 0 & -3.33 & 3.33 \end{bmatrix}. \quad (9c)$$

We will show that the electrical circuit described by the equation (9a) with matrices (9b) and (9c) is reachable for constant inputs  $U = [E_1 \ E_2 \ E_3]^T$  in time  $t_f = 5s$ .

Using Theorem 1 we have

$$R_c = A^{-1} (e^{At_f} - \mathbb{I}_n) B = \begin{bmatrix} 0.91 & -0.91 & 0 \\ -0.91 & -0.09 & 1 \end{bmatrix} \quad (10)$$

and the condition (5) is met, since

$$\text{rank} R_c = n = 2. \quad (11)$$

Therefore, there exists constant input vector  $U$  that steers the circuit from zero initial conditions  $x_0 = [i_L(0) \ u_C(0)]^T = [0 \ 0]^T$  to the arbitrary given final state  $x_f = x(t_f) = [i_L(t_f) \ u_C(t_f)]^T$  in desired time  $t_f = 5s$ .

Let  $i_L(t_f) = i_L(5) = 1A$ ,  $u_C(t_f) = u_C(5) = 0.5V$ . Then the constant inputs can be computed using (7) and (8a) with  $K_1 = 0$

$$U = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = R_c^T [R_c R_c^T]^{-1} x_f = \begin{bmatrix} 0.4 & -0.33 \\ -0.7 & -0.33 \\ 0.3 & 0.67 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.24 \\ -0.87 \\ 0.64 \end{bmatrix}. \quad (12)$$

The state variables of the electrical circuit shown in Fig. 1 with constant inputs (12) are shown in Fig. 2.

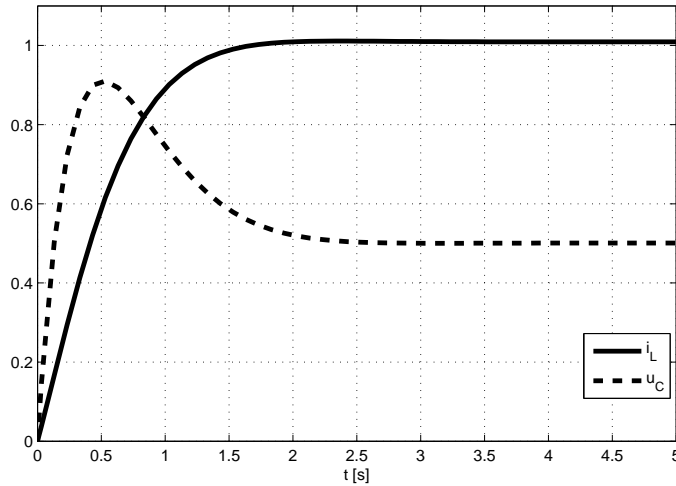


Figure 2. State variables of Example 1.

### 3. Reachability of fractional continuous-time systems with constant inputs

Let us consider the fractional continuous-time system described by the state equation [13, 16]

$$D^\alpha x(t) = Ax(t) + Bu(t), \quad (13)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $u(t) \in \mathbb{R}^m$  is the input vector of the system for  $t \geq 0$ , matrices  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  and

$$D^\alpha f(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^t (t-\tau)^{N-\alpha-1} \frac{d^N f(\tau)}{d\tau^N} d\tau \quad (14)$$

is the  $\alpha$ -order ( $\alpha \in \mathbb{R}$ ) fractional derivative described by the Caputo operator, where  $N-1 \leq \alpha < N$ ,  $N \in \mathbb{N}$  and  $\Gamma(x)$  is the Euler gamma function.

The solution to the state equation (13) with initial condition  $x_0 = x(0) \in \mathbb{R}^n$  and input vector  $u(t) \in \mathbb{R}^m$  for  $t \geq 0$  is given by [13, 16]

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau = \Phi_0(t)x_0 + \int_0^t \Phi(\tau)Bu(t-\tau)d\tau, \quad (15)$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} = E_\alpha(At^\alpha), \quad (16a)$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha). \quad (16b)$$

and  $E_\alpha(x)$ ,  $E_{\alpha,\beta}(x)$  are the one and two parameters Mittag-Leffler functions, respectively [13, 16].

Now we define the notion of reachability of the system described by the state equation (13) for constant inputs (3).

**Definition 2** *The fractional continuous-time system (13) is called reachable in time  $t_f > 0$  if there exists a constant input vector (3) that steers the system from zero initial conditions  $x_0 = x(0) = 0$  to every given final state  $x_f = x(t_f) \in \mathbb{R}^n$ .*

**Theorem 2** *The fractional continuous-time system (13) is reachable for constant inputs in time  $t_f > 0$  if and only if the matrix*

$$R_f = E_{\alpha,\alpha+1}(At_f^\alpha)B \in \mathbb{R}^{n \times n} \quad (17)$$

has  $n$  linearly independent columns, i.e.

$$\text{rank}R_f = n. \quad (18)$$

**Proof** Substituting (3) and (16b) into (15) for zero initial conditions and  $t = t_f > 0$  we obtain

$$\begin{aligned} x(t_f) = x_f &= \int_0^{t_f} \Phi(\tau)BU \, d\tau = \int_0^{t_f} \Phi(\tau) \, d\tau BU = \int_0^{t_f} \sum_{k=0}^{\infty} \frac{A^k \tau^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \, d\tau BU \\ &= \sum_{k=0}^{\infty} \frac{A^k}{\Gamma[(k+1)\alpha]} \int_0^{t_f} \tau^{(k+1)\alpha-1} \, d\tau BU = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma[(k+1)\alpha+1]} \left[ \tau^{(k+1)\alpha} \right]_0^{t_f} BU \quad (19) \\ &= \sum_{k=0}^{\infty} \frac{A^k}{\Gamma[(k+1)\alpha+1]} t_f^{(k+1)\alpha} BU = t_f^\alpha E_{\alpha,\alpha+1}(At_f^\alpha)BU, \end{aligned}$$

since from the properties of the gamma function we have  $[(k+1)\alpha]\Gamma[(k+1)\alpha] = \Gamma[(k+1)\alpha+1]$ .

From (19) it follows that the equation

$$x_f = t_f^\alpha E_{\alpha,\alpha+1}(At_f^\alpha)BU \quad (20)$$

has a solution  $U$  for given final state  $x_f$  and given final time  $t_f > 0$  if and only if the condition (18) is satisfied.  $\square$

The constant input that steers the system (13) from zero initial state to the final state  $x_f$  in time  $t \in [0, t_f]$  is given by the formula

$$U = t_f^{-\alpha} R_f^+ x_f, \quad (21)$$

where  $R_f^+ \in \mathbb{R}^{m \times n}$  is the right pseudoinverse of the rectangular matrix  $R_f$  given by one of the following equalities [6, 9]

$$R_f^+ = R_f^T [R_f R_f^T]^{-1} + \left( \mathbb{I}_m - R_f^T [R_f R_f^T]^{-1} R_f \right) K_1 \quad \text{for arbitrary } K_1 \in \mathbb{R}^{m \times n}; \quad (22a)$$

$$R_f^+ = K_2 [R_f K_2]^{-1} \quad \text{for arbitrary } K_2 \in \mathbb{R}^{m \times n}, \det[R_f K_2] \neq 0. \quad (22b)$$

From Theorems 1 and 2 we have the following corollary.

**Corollary 1** *The system (1) (or (13)) is unreachable for constant inputs if the number of inputs is less than the number of state variables, i.e.  $m < n$ .*

**Example 2** Consider the electrical circuit from Example 1 shown in Fig. 1 with  $\alpha = 0.8$ .

Using Kirchhoff's laws for the fractional electrical circuit shown in Fig. 1, it can be easily shown, that the state equation of this circuit has the form

$$\frac{d^\alpha}{dt^\alpha} \begin{bmatrix} i_L(t) \\ u_C(t) \end{bmatrix} = A \begin{bmatrix} i_L(t) \\ u_C(t) \end{bmatrix} + B \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} \quad (23)$$

with the matrices given by (9b) and (9c).

We will show that the fractional electrical circuit is reachable for constant input vector  $U = [E_1 \ E_2 \ E_3]^T$  in time  $t_f = 5\text{s}$ .

Using Theorem 2 we have

$$R_f = E_{\alpha, \alpha+1}(A t_f^\alpha) B = \begin{bmatrix} 0.24 & -0.24 & 0 \\ -0.24 & -0.034 & 0.27 \end{bmatrix} \quad (24)$$

and the condition (18) is met, since

$$\text{rank} R_f = n = 2. \quad (25)$$

Therefore, there exists constant input vector  $U$  that steers the circuit from zero initial conditions  $x_0 = [i_L(0) \ u_C(0)]^T = [0 \ 0]^T$  to the arbitrary given final state  $x_f = x(t_f) = [i_L(t_f) \ u_C(t_f)]^T$  in desired time  $t_f = 5\text{s}$ .

Let  $i_L(t_f) = i_L(5) = 1\text{A}$ ,  $u_C(t_f) = u_C(5) = 0.5\text{V}$ . Then the constant inputs can be computed using (21) and (22a) with  $K_1 = 0$

$$U = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = t_f^{-\alpha} R_f^T [R_f R_f^T]^{-1} x_f = 5^{-0.8} \begin{bmatrix} 1.55 & -1.24 \\ -2.62 & -1.24 \\ 1.05 & 2.44 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.26 \\ -0.89 \\ 0.63 \end{bmatrix}. \quad (26)$$

The state variables of the fractional electrical circuit with constant voltage sources (26) are shown in Fig. 3.

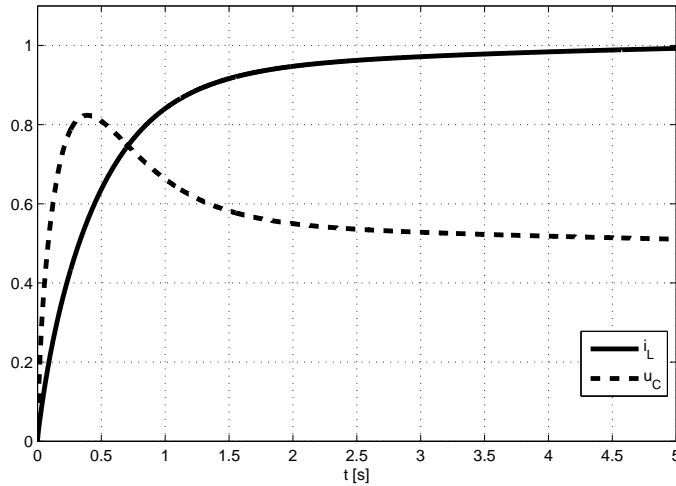


Figure 3. State variables of Example 2.

#### 4. Reachability of positive systems with constant inputs

**Definition 3** The system (1) (or fractional system (13) for  $0 < \alpha < 1$ ) is called positive if the state vector  $x(t) \in \mathbb{R}_+^n$ ,  $t > 0$  for all initial conditions  $x_0 \in \mathbb{R}_+^n$  and all nonnegative inputs  $u(t) \in \mathbb{R}_+^m$ ,  $t \geq 0$ .

**Theorem 3** [5, 13, 16] The continuous-time system (1) (or fractional system (13) for  $0 < \alpha < 1$ ) is positive if and only if

$$A \in M_n, \quad B \in \mathbb{R}_+^{n \times m}. \quad (27)$$

**Theorem 4** The standard positive continuous-time system (1) is reachable for constant inputs in time  $t_f > 0$  if and only if the matrix  $A$  is diagonal and  $B$  is monomial.

**Proof** From (7) it follows that  $U \in \mathbb{R}_+^m$  if and only if  $R_c^+ \in \mathbb{R}_+^{m \times n}$ , since  $x_f \in \mathbb{R}_+^n$ . It is easy to show that the right pseudoinverse has all nonnegative elements if and only if the matrix  $R_c$  has  $n$  monomial rows.

From (4), for monomial matrix  $B$ , we obtain the necessity of monomiality of the matrix

$$A^{-1} (e^{At_f} - \mathbb{I}_n). \quad (28)$$

The matrix (28) is monomial if and only if the matrix  $A$  is diagonal. Moreover the matrix has all nonnegative elements for arbitrary diagonal entries of matrix  $A = \text{diag}(a_{11}, a_{22}, \dots, a_{nn})$ , since

$$A^{-1} (e^{At_f} - \mathbb{I}_n) = \text{diag} [a_{11}^{-1}(e^{a_{11}t_f} - 1), a_{22}^{-1}(e^{a_{22}t_f} - 1), \dots, a_{nn}^{-1}(e^{a_{nn}t_f} - 1)] \quad (29)$$

is nonnegative diagonal matrix for arbitrary  $a_{11}, a_{22}, \dots, a_{nn}$ . □



**Theorem 5** *The fractional positive continuous-time system (13) is reachable for constant inputs in time  $t_f > 0$  if and only if the matrix  $A$  is diagonal and  $B$  is monomial.*

The proof of this theorem is similar to the proof of Theorem 4.

**Corollary 2** *Note that the conditions for reachability for constant inputs of positive standard and fractional systems are the same as for the reachability for arbitrary (non-constant) continuous-time inputs (see [16]).*

**Example 3** Consider the electrical circuit shown in Fig. 4 with given resistances  $R_1 = 0.5\Omega$ ,  $R_2 = 1\Omega$ , inductances  $L_1 = 0.5\text{H}$ ,  $L_2 = 0.4\text{H}$  and source volages  $e_1$ ,  $e_2$ .

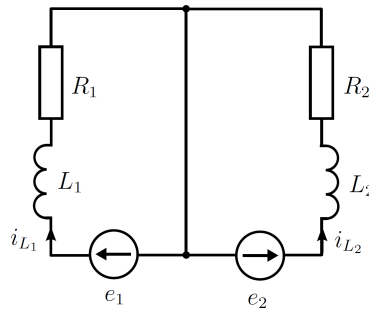


Figure 4. Electrical circuit of Example 3.

Using Kirchhoffs laws we may formulate the state equations (1) of the circuit shown in Fig. 4

$$\frac{d}{dt} \begin{bmatrix} i_{L_1}(t) \\ i_{L_2}(t) \end{bmatrix} = A \begin{bmatrix} i_{L_1}(t) \\ i_{L_2}(t) \end{bmatrix} + B \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}, \quad (30a)$$

where

$$A = \begin{bmatrix} -\frac{R_1}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (30b)$$

By Theorem 3, the electrical circuit described by the state equation (30a) with matrices (30b) is positive.

We will show that this electrical circuit is reachable for constant inputs vector  $U = [E_1 \ E_2]^T$  in time  $t_f = 3\text{s}$ .

The conditions of Theorem 4 are met. The matrix

$$R_c = A^{-1} (e^{At_f} - \mathbb{I}_n) B = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.4 \end{bmatrix} \quad (31)$$

has  $n$  monomial rows. Therefore, there exists constant input vector  $U$  that steers the circuit from zero initial conditions  $x_0 = [i_{L_1}(0) \ i_{L_2}(0)]^T = [0 \ 0]^T$  to the arbitrary given final state  $x_f = x(t_f) = [i_{L_1}(t_f) \ i_{L_2}(t_f)]^T$  in time  $t_f = 3s$ .

Let  $i_{L_1}(t_f) = i_{L_1}(3) = 1A$ ,  $i_{L_2}(t_f) = i_{L_2}(3) = 1.5A$ . Then the constant nonnegative inputs vector can be computed using (7)

$$U = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = R_c^T [R_c R_c^T]^{-1} x_f = \begin{bmatrix} 1.05 & 0 \\ 0 & 2.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 1.05 \\ 3.75 \end{bmatrix}. \quad (32)$$

The state variables of the electrical circuit shown in Fig. 4 with constant inputs (32) are shown in Fig. 5.

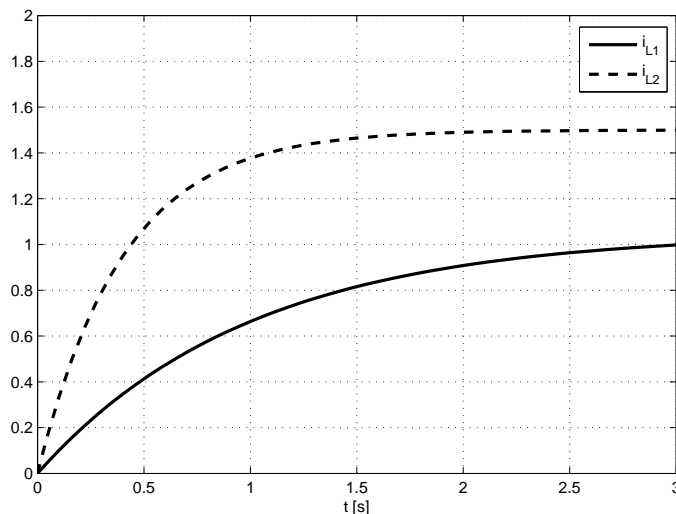


Figure 5. State variables of Example 3.

**Example 4** Consider the fractional electrical circuit from Example 3 shown in Fig. 4 with  $\alpha = 0.8$ .

Using Kirchhoff's laws for the fractional electrical circuit we may write the state equation

$$\frac{d^\alpha}{dt^\alpha} \begin{bmatrix} i_{L_1}(t) \\ i_{L_2}(t) \end{bmatrix} = A \begin{bmatrix} i_{L_1}(t) \\ i_{L_2}(t) \end{bmatrix} + B \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \quad (33)$$

with the matrices given by (30b).

We will show that the fractional electrical circuit is reachable for constant input vector  $U = [E_1 \ E_2]^T$  in time  $t_f = 3s$ .

From Theorem 5 follows that the fractional circuit is reachable for constant inputs, since the matrix  $A$  is diagonal matrix and  $B$  is monomial.

Using (17) we have

$$R_f = E_{\alpha, \alpha+1}(A t_f^\alpha) B = \begin{bmatrix} 0.35 & 0 \\ 0 & -0.16 \end{bmatrix} \quad (34)$$

and the matrix  $R_f$  has two monomial rows.

Therefore, there exists nonnegative constant input vector  $U$  that steers the fractional electrical circuit from zero initial conditions  $x_0 = [i_{L_1}(0) \ i_{L_2}(0)]^T = [0 \ 0]^T$  to the arbitrary given final state  $x_f = x(t_f) = [i_{L_1}(t_f) \ i_{L_2}(t_f)]^T$  in desired time  $t_f = 3$ s.

Let  $i_{L_1}(t_f) = i_{L_1}(3) = 1$ A,  $i_{L_2}(t_f) = i_{L_2}(3) = 1.5$ A. Then the nonnegative constant inputs can be computed using (21) and (22a) with  $K_1 = 0$

$$U = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = t_f^{-\alpha} R_f^T [R_f R_f^T]^{-1} x_f = 3^{-0.8} \begin{bmatrix} 2.86 & 0 \\ 0 & 6.25 \end{bmatrix} \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 1.19 \\ 3.89 \end{bmatrix}. \quad (35)$$

The state variables of the fractional electrical circuit shown in Fig. 4 with constant voltage sources (35) are shown in Fig. 6.

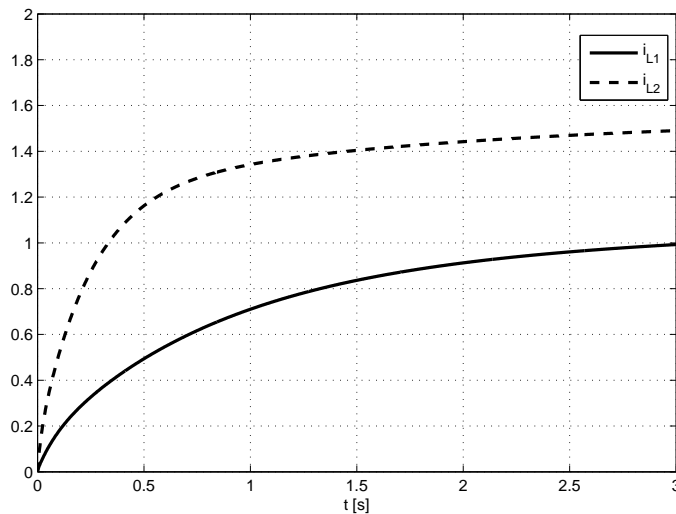


Figure 6. State variables of Example 4.

## 5. Concluding remarks

The reachability of standard (ordinary-order) and fractional-order systems with constant inputs has been considered. Necessary and sufficient conditions for non-positive and positive continuous-time systems have been established (Theorem 1-5). It has been shown that the conditions for reachability of nonpositive and positive fractional-order systems for constant inputs and arbitrary (nonconstant) inputs are the same (Corollary 2). The considerations are illustrated by examples of electrical circuits with DC voltage sources.

## References

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