10.2478/acsc-2014-0015

Archives of Control Sciences
Volume 24(LX), 2014
No. 3, *pages 257–270*

Pinning synchronization of the drive and response dynamical networks with lag

BOHUI WEN, MO ZHAO and FANYU MENG

This paper investigates the pinning synchronization of two general complex dynamical networks with lag. The coupling configuration matrices in the two networks are not need to be symmetric or irreducible. Several convenient and useful criteria for lag synchronization are obtained based on the lemma of Schur complement and the Lyapunov stability theory. Especially, the minimum number of controllers in pinning control can be easily obtained. At last, numerical simulations are provided to verify the effectiveness of the criteria.

Key words: complex networks, lag synchronization, pinning control, Lorenz system

1. Introduction

It is universally acknowledged that there are many complex networks in various fields of the real world, such as social networks, biological networks [1, 2], world-wide web [3], and food webs [4]. In the past few years, several kinds of network models have been proposed for the purpose of describing the real world more realistic [1, 5, 6, 7, 8]. In the complex dynamical networks, one of the most remarkable phenomena is their spontaneous synchronization, and so many types of synchronization, such as complete synchronization [9], phase synchronization [10], projective synchronization [11, 12, 13], impulsive synchronization [14, 15, 16], and cluster synchronization [17, 18, 19] have deeply caught the eyes of the researchers in the past few decades. However, the phenomenon of synchronization also can be classified into 'inner synchronization' [14, 20] and 'outer synchronization' [21, 22] from another point of view. 'Inner synchronization' means a collective behavior within a network, while 'outer synchronization' refers to the synchronization occurring between two or more coupled networks regardless of happening of the inner synchronization. In nature, many phenomena can explain the phenomenon of outer synchronization, such as the infectious disease spreads between

Bohui Wen, the corresponding author, is with School of Finance, Tianjin University of Finance & Economics, Tianjin, 300222, China, e-mail: mimiwenh@163.com. Mo Zhao is with Civil Aviation University of China, 300300, Tainjin, China, e-mail: zhaomo2008@163.com. Fanyu Meng is with Basic Experimental Center, Civil Aviation University of China, Tianjin, 300222, China, e-mail: fymeng@yeah.net.

This work is supported by the National Natural Science Foundation of China (71103126).
Received 9.07.2014.

different communities, the avian influenza spreads among domestic and wild birds, and the different species development in balance. All these challenging topics show the great importance of researching the outer synchronization between coupled networks.

In the process of researching inner synchronization, time delay need to be considered because of the network congestion as well as the finite speed of signal transmission over the links. Nevertheless, the phenomenon of time delay also occurs in the process of outer synchronization and effects the dynamical behaviors of nodes in the two networks. Usually, this kind of time delay is named as lag. Lag synchronization requires the states of response system to synchronize with the past states of the drive system, and a common example is the telephone communication system, in which the voice one hears on the receiver side at time $t+\tau$ is the voice from the transmitter side at time t [23]. Therefore, it is necessary to take into account the impact of lag during the process of modeling. Pinning control, which was first presented by Hu [24] in 1999, is a useful method by adding controllers to part nodes in the network to ensure the whole network achieve synchronization [25]. Because of the complexity of complex networks, the method of pinning control not only simplifies the coupling topology configuration, but also saves the costs [20, 21]. Therefore, pinning control has become a hotspot for its advantages.

Motivated by the above discussions, we mainly investigate the lag synchronization of complex networks via pinning control. Compared with the method proposed by Guo in [26], there is no need to consider the reducibility of coupled matrix G for obtaining the criteria of lag synchronization, and there is also no extra constraint imposed on the inner coupling matrix. Some sufficient conditions for lag synchronization are achieved by using the Lyapunov stability theorem and adaptive techniques. In addition, we could obtain the minimum number of controllers in pinning control easily.

The rest of this paper is organized as follows. In Section 2, some general driver and response complex dynamical network models are introduced, and some necessary preliminaries are given. In Section 3, based on the Lyapunov stability theorem, some pinning controllers are designed to ensure the driver and response systems achieve outer synchronization under different situations. In Section 4, some numerical simulations are given to verify the effectiveness of proposed theoretical results and the Section 5 is the conclusion of the paper.

2. Network model and preliminaries

In this section, a general complex dynamical network consisting of N identical nodes with linearly diffusive coupling will be introduced, and the state equation of the drive complex dynamical network is described by

$$\dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{N} g_{ij}Ax_j(t), \quad i = 1,2,\cdots,N. \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in R^n$ is the state variable of the ith node, $f(x_i) : R^n \longrightarrow R^n$ is a nonlinear vector valued function describing the dynamics of nodes and $c > 0$ is the coupling strength of the whole network. The matrix $G = (g_{ij}) \in R^{N\times N}$ is the outer coupling configuration matrix, in which $g_{ij} \in R$ is defined as follows: if there is a coupling from node i to node j $(i \neq j)$, $g_{ij} > 0$; otherwise, $g_{ij} = 0$. At the same time, the diagonal elements of G are defined as $g_{ii} = -\sum_{j=1,j\neq i}^{N} g_{ij}$. The inner coupling matrix A denotes the inner coupling relation between every two nodes.

Compared with the drive system mentioned above, the response complex network with pining controllers is designed as

$$\begin{cases} \dot{y}_i(t) = f(y_i(t)) + c\sum\limits_{j=1}^{N} g_{ij}Ay_j(t) + u_i(t), & i = 1,2,\cdots,l \\ \dot{y}_i(t) = f(y_i(t)) + c\sum\limits_{j=1}^{N} g_{ij}Ay_j(t), & i = l+1,\cdots,N. \end{cases} \tag{2}$$

where $y_i(t) = (y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t))^T \in R^n$ denotes the state variables of the response system, and other parameters involved in the system (2) all have the same meanings with the corresponding parameters in system (1).

Remark 1 In these two system models mentioned above, the outer coupling configuration matrix G does not need to be symmetric or irreducible.

In the following, some needed definition, assumption and lemmas will be presented.

Definition 1 [27] *The drive system* (1) *and the response system* (2) *are defined to achieve lag synchronization at time* τ, *if*

$$y_i(t) - x_i(t-\tau) \to 0\,, \quad t \to \infty\,, \quad i = 1\,,\cdots\,,N. \tag{3}$$

where τ *is a positive time-delay.*

Lemma 1 *(Schur complement)* [28] *The following linear matrix inequality (LMI)*

$$\begin{pmatrix} \mathcal{A}(x) & \mathcal{B}(x) \\ (\mathcal{B}(x))^T & \mathcal{C}(x) \end{pmatrix} > 0,$$

where $\mathcal{A}(x) = (A(x))^T, \mathcal{C}(x) = (C(x))^T$ *is equivalent to one of the following conditions:*

(a) $\mathcal{A}(x) > 0$ *and* $\mathcal{C}(x) - \mathcal{B}(x)^T\mathcal{A}(x)^{-1}\mathcal{B}(x) > 0$;

(b) $\mathcal{C}(x) > 0$ *and* $\mathcal{A}(x) - \mathcal{B}(x)\mathcal{C}(x)^{-1}\mathcal{B}(x)^T > 0$.

Lemma 2 [29] *Assume that* $\bar{A} = \begin{pmatrix} A_1 & A_3 \\ A_3^T & A_2 \end{pmatrix}, \bar{B} = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}$, *where* $\bar{A}, \bar{B} \in R^{N\times N}$, $A_1, B_1 \in R^{r\times r}(1 \leqslant r < N), B_1 = diag\{b_1, \ldots, b_r\}$ *ia a diagonal positive matrix,* $A_1^T = A_1$, *and* $A_2^T = A_2$. *Then* $\bar{A} - \bar{B} < 0$ *is equivalent to* $A_2 < 0$ *for large enough* $b_i(1 \leqslant i \leqslant r)$.

Throughout this paper, the following assumption will be required and the sign $\| * \|$ represents the Euclid norm.

Assumption 1 [30] *The nonlinear function $f(\cdot)$ satisfies the following Lipschitz condition:*

$$\|f(y(t)) - f(x(t))\| \leqslant \delta \| (y(t) - x(t)) \|, \ \forall\, x\,,\, y \in R^n, \tag{4}$$

where δ is a known positive constant.

It has been verified that many typical chaotic systems such as Lorenz system, Chen system, Lü system, and the unified chaotic system all satisfy the above assumption.

3. Main Results

In this part, two kinds of pinning controllers will be designed to make the system (1) and system (2) achieve synchronization with lag.

Define error vectors as

$$e_i(t) = y_i(t) - x_i(t-\tau), \qquad i = 1, 2, ..., N, \tag{5}$$

where $\tau > 0$ denotes a lag between the drive and response systems.

According to (1), (2) and (5) , the following equations would be acquired

$$\begin{cases} \dot{e}_i(t) = f(y_i(t)) - f(x_i(t-\tau)) + c\sum\limits_{j=1}^{N} g_{ij}Ae_j(t) + u_i(t), \ i = 1, 2, \cdots, l \\ \dot{e}_i(t) = f(y_i(t)) - f(x_i(t-\tau)) + c\sum\limits_{j=1}^{N} g_{ij}Ae_j(t), \qquad i = l+1, \cdots, N. \end{cases} \tag{6}$$

In this paper, we choose $\hat{G}$ as a modifying matrix of G via replacing the diagonal elements g_{ii} by $(\rho_{\min}/\gamma)g_{ii}$, where $\gamma = \|A\|$ and $\rho_{\min}$ denotes the maximum eigenvalue of matrix $\frac{A+A^T}{2}$. The matrix $\left(\frac{\hat{G}+\hat{G}^T}{2}\right)_{(N-l)(N-l)}$ is the principal submatrix of $\frac{\hat{G}+\hat{G}^T}{2}$ by removing the first l rows and columns, and $\lambda_{(N-l)}$ denotes the maximum eigenvalue of matrix $\left(\frac{\hat{G}+\hat{G}^T}{2}\right)_{(N-l)(N-l)}$.

3.1. Pinning Control of the Lag Synchronization

In this section, several sufficient conditions for lag synchronization will be provided by adopting different control schemes.

First, some simple state feedback pinning controllers will be designed as follows:

$$\begin{cases} u_i(t) = -k_i e_i(t), & i = 1, 2, \cdots, l \\ u_i(t) = 0, & i = l+1, \cdots, N. \end{cases} \tag{7}$$

where $k_i > 0$ denotes the feedback gain.

Theorem 2 *Suppose the Assumption* 1 *holds, if there exists a natural number* $1 \leqslant l < N$ *satisfying* $\lambda_{(N-l)} < -\frac{\delta}{c\gamma}$, *and the positive constant* $k = min(k_1, k_2, \dots, k_l)$ *is large enough, the drive network* (1) *and the response network* (2) *can realize lag synchronization with the controllers* (7).

Proof Construct a Lyapunov function as follows:

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i{}^T(t)\, e_i(t), \tag{8}$$

Differentiating $V(t)$ with respect to time along the solution of (6) yields

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{N} e_i^T(t)\dot{e}_i(t) \\
&= \sum_{i=1}^{N} e_i^T(t)\left[f(y_i(t)) - f(x_i(t-\tau)) + c\sum_{j=1}^{N} g_{ij}Ae_j(t)\right] - \sum_{i=1}^{l} e_i^T(t)k_i e_i(t) \\
&\leqslant \sum_{i=1}^{N} e_i^T(t)\delta e_i(t) + c\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} g_{ij}e_i^T(t)Ae_j(t) \\
&\quad + c\sum_{i=1}^{N} g_{ii}e_i^T(t)\frac{A+A^T}{2}e_i(t) - \sum_{i=1}^{l} e_i^T(t)k_i e_i(t) \\
&\leqslant \sum_{i=1}^{N} \delta e_i^T(t)e_i(t) + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} c\gamma g_{ij}\left\|e_i(t)\right\|\left\|e_j(t)\right\| \\
&\quad + \sum_{i=1}^{N} cg_{ii}\rho_{\min}e_i^T(t)e_i(t) - \sum_{i=1}^{l} e_i^T(t)k_i e_i(t) \\
&= \mathbf{e}^T(\delta I_N + c\gamma\hat{G} - K)\mathbf{e} \\
&= \mathbf{e}^T\left(\delta I_N + c\gamma\frac{\hat{G}+\hat{G}^T}{2} - K\right)\mathbf{e},
\end{aligned} \tag{9}$$

where

$$\mathbf{e} = (\|e_1(t)\|, \|e_2(t)\|, \dots, \|e_N(t)\|)^T,$$

$$K = \mathrm{diag}(\underbrace{k_1, \quad \dots, \quad k_l}_{l}, \underbrace{0, \quad \dots, \quad 0}_{N-l}),$$

and I_N is an $N \times N$ dimensional identity matrix. Because of the symmetry of matrix $(\hat{G} + \hat{G}^T)/2$ and Lemma 2, $\delta I_N + c\gamma\frac{\hat{G}+\hat{G}^T}{2} - K < 0$ is equivalent to $\delta \mathrm{I}_{N-l} +$

$c\gamma\left(\frac{\hat{G}+\hat{G}^T}{2}\right)_{(N-l)(N-l)} < 0$ for large enough $k_i, i = 1,2,\ldots,l$. Thus, with the conditions in Theorem 1, it is clear that if there exists a natural number $1 \leqslant l < N$ satisfying $\delta + c\gamma\lambda_{(N-l)} < 0$, the matrix inequality $\delta \mathrm{I}_{N-l} + c\gamma\left(\frac{\hat{G}+\hat{G}^T}{2}\right)_{(N-l)(N-l)} < 0$ will hold, where l denotes the number of nodes needed to add controllers. Therefore, when the parameter $k = \min(k_1,k_2,\ldots,k_l)$ is chosen large enough and the inequality $\lambda_{(N-l)} < -\frac{\delta}{c\gamma}$ holds, the drive system can achieve lag synchronization with the response system by using the controllers in (7). Then, the proof is completed. □

From the conclusion in Theorem 1, if the response system with linear state feedback controllers in (7) intend to synchronize with the drive system at time τ, the controller gain should be large enough. However, in the real applications, to obtain a high gain is usually at the expense of system's stability and may generate a phenomenon of turbulence. Thus, it is a good idea to design some adaptive pinning controllers to make the drive and response systems achieve lag synchronization.

Theorem 3 *Under the Assumption* 1 *, if there exists a natural number* $1 \leqslant l < N$ *satisfying* $\lambda_{(N-l)} < -\frac{\delta}{c\gamma}$, *the drive network* (1) *can synchronize with the response system* (2) *at time* τ *by using the following adaptive controllers:*

$$\begin{cases} u_i(t) = -p_i(t)e_i(t), \quad \dot{p}_i(t) = q_i e_i{}^2(t), \quad i = 1,2,\cdots,l \\ u_i(t) = 0, \quad\quad\quad\quad\quad\quad\quad i = l+1,\cdots,N. \end{cases} \tag{10}$$

where q_i *is a positive constant.*

Proof Define a Lyapurov function as follows:

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t) + \frac{1}{2}\sum_{i=1}^{l}\frac{(p_i(t)-p)^2}{q_i}. \tag{11}$$

where p is a sufficient large positive constant to be determined.

The derivative of $V(t)$ along the trajectory (6) is

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{N} e_i^T(t)\dot{e}_i(t) + \sum_{i=1}^{l}\frac{1}{q_i}(p_i(t)-p)\dot{p}_i(t) \\ &= \sum_{i=1}^{N} e_i^T(t)[f(y_i(t)) - f(x_i(t-\tau))] + c\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)g_{ij}Ae_j(t) \\ &\quad - \sum_{i=1}^{l} e_i^T(t)p_i(t)e_i(t) + \sum_{i=1}^{l}(p_i(t)-p)\|e_i(t)\|^2 \\ &\leqslant \sum_{i=1}^{N} \delta e_i^T(t)e_i(t) + c\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} e_i^T(t)g_{ij}Ae_j(t) \end{aligned}$$

$$
\begin{aligned}
& + c\sum_{i=1}^{N} g_{ii} e_i^T(t)\frac{A+A^T}{2}e_i(t) - \sum_{i=1}^{l} p e_i^T(t)e_i(t) \\
\leqslant\ & \sum_{i=1}^{N} \delta e_i^T(t)e_i(t) + c\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} \gamma g_{ij}\|e_i(t)\|\|e_j(t)\| \\
& + c\sum_{i=1}^{N} g_{ii}\rho_{\min} e_i^T(t)e_i(t) - \sum_{i=1}^{l} p e_i^T(t)e_i(t) \\
=\ & \mathbf{e}^T\left(\delta I_N + c\gamma\frac{\hat{G}+\hat{G}^T}{2} - D\right)\mathbf{e},
\end{aligned} \tag{12}
$$

where $D = \mathrm{diag}\{\underbrace{p,\dots,p}_{l},\underbrace{0,\dots,0}_{N-l}\}$, $\mathbf{e} = (\|\mathbf{e}_1(t)\|,\|\mathbf{e}_2(t)\|,\dots,\|\mathbf{e}_N(t)\|)^T$.

According to Lemma (2), $\delta \mathrm{I}_N + c\gamma\frac{\hat{G}+\hat{G}^T}{2} - D < 0$ is equivalent to $\delta \mathrm{I}_{N-l} + c\gamma(\frac{\hat{G}+\hat{G}^T}{2})_{(N-l)(N-l)} < 0$ for a large enough p. However, from the Theorem 1, it has been known that if there exists a natural number $1 \leqslant l < N$ satisfying $\delta + c\gamma\lambda_{(N-l)} < 0$, the inequality $\delta \mathrm{I}_{N-l} + c\gamma(\frac{\hat{G}+\hat{G}^T}{2})_{(N-l)(N-l)} < 0$ could be established, where l denotes the number of nodes needed to add controllers. Therefore, with the conditions in Theorem 2 and a large enough positive constant p, the inequality $\delta \mathrm{I}_N + \gamma\frac{\hat{G}+\hat{G}^T}{2} - D < 0$ holds. That is, the error vector $\mathbf{E} = (\mathbf{e}_1^T,\mathbf{e}_2^T,\dots,\mathbf{e}_N^T)^T \to 0$ as $t \to +\infty$. Then, the proof is thus completed. $\square$

Remark 2 For resolving the problem of lag synchronization, Guo gave two useful theorems in [26] while this paper provides two simpler criteria without considering the reducibility or irreducibility of the matrix G.

Remark 3 From the Theorem 2, it not only can judge whether the two networks achieve lag synchronization, but also know the minimum number of controllers needed. In addition, the inequality $\lambda_{(N-l)} < -\frac{\delta}{c\gamma}$ in Theorem 2 reflects the relationship between the number of controllers l and the coupling strength c. Base on this relationship, it is easy to change the number of nodes to be pinned by adjusting the coupling strength c.

4. Numerical Simulations

In this part, various numerical simulations are presented to verify the effectiveness of the proposed criteria. Select a drive system with six nodes and so does the response system. The nodes in the drive and response systems are all chosen as Lorenz system, which is described as follows:

$$\begin{cases} \dot{x_1}(t) = a(x_2(t) - x_1(t)) \\ \dot{x_2}(t) = cx_1(t) - x_1(t)x_3(t) - x_2(t) \\ \dot{x_3}(t) = x_1(t)x_2(t) - bx_3(t), \end{cases} \tag{13}$$

where $a = 10, b = 8/3, c = 28$.

Because of the boundedness of the Lorenz system showed in Fig. 1, the Assumption 1 is satisfied and the parameter can be chosen as $\delta = 60.318$.

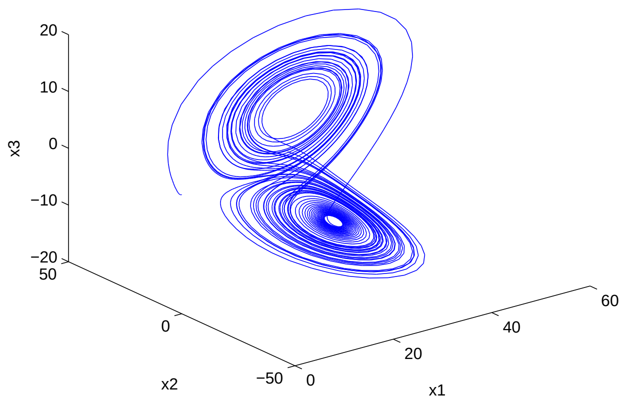

Figure 1. The chaotic attractor of the Lorenz system.

Example 1. In this example, some adaptive pinning controllers will be designed with the coupling strength $c = 40$. Meanwhile, choose the outer coupling matrix as

$$G = \begin{bmatrix} -6 & 1 & 2 & 1 & 1 & 1 \\ 1 & -5 & 2 & 1 & 0 & 1 \\ 2 & 2 & -7 & 0 & 1 & 2 \\ 1 & 1 & 0 & -7 & 2 & 3 \\ 1 & 0 & 1 & 2 & -5 & 1 \\ 1 & 1 & 2 & 3 & 1 & -8 \end{bmatrix}. \tag{14}$$

which is symmetric and irreducible. The inner coupling matrix is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}. \tag{15}$$

With the help of Matlab, it is easy to calculate the maximum eigenvalue of the principal submatrix with different orders. Then, according to Theorem 2, the number of nodes to be pinned will be obtained, and this result is more accurate than the one obtained in [20] and [21]. After some operations with the parameters given above, only one controller is needed to make the two networks achieve lag synchronization. Without loss of generality, one node in the response system will be selected as the pinning node randomly. Compared with the results obtained in [26], less controllers are needed to make the two systems achieve lag synchronization, and this demonstrates that the method in this paper is more practical and economical in engineering applications. Select the initial values of $x(0), y(0)$ and $p(0)$ in $(0,1)$ randomly, then the error variables of lag synchronization with $\tau = 0.5$ are shown in Fig. 2, Fig. 3 and Fig. 4.

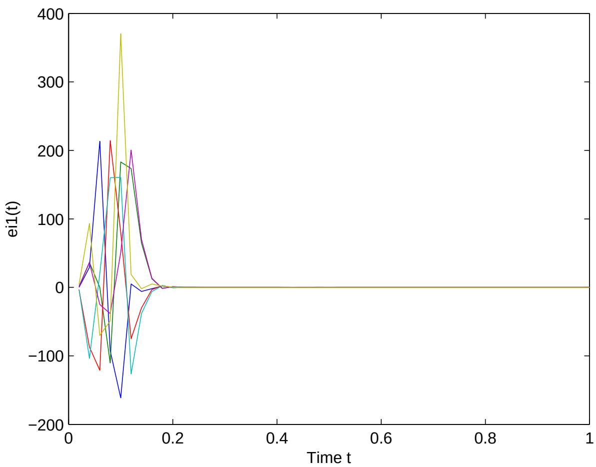

Figure 2. The lag synchronization error $e_{i1}, (1 \leqslant i \leqslant 6)$ between the driver and response networks.

Example 2. In this example, the coupling configuration matrix is chosen with the characters of reducibility and asymmetry as follows:

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 & 0 \\ 1 & 1 & 0 & -7 & 2 & 3 \\ 1 & 0 & 1 & 2 & -5 & 1 \\ 1 & 1 & 2 & 3 & 1 & -8 \end{bmatrix}. \tag{16}$$

other parameters and the initial values of the state variables in the drive and response systems are selected as the same as in Example 1. After operations, two controllers are needed to add on the nodes in the response system. Then, the error variables of the lag

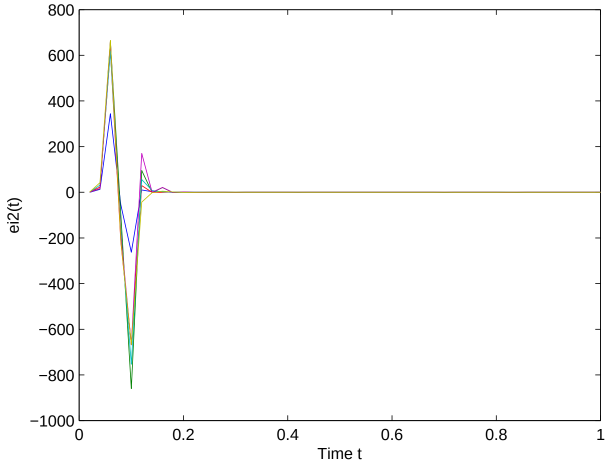

Figure 3. The lag synchronization error $e_{i2}, (1 \leqslant i \leqslant 6)$ between the driver and response networks.

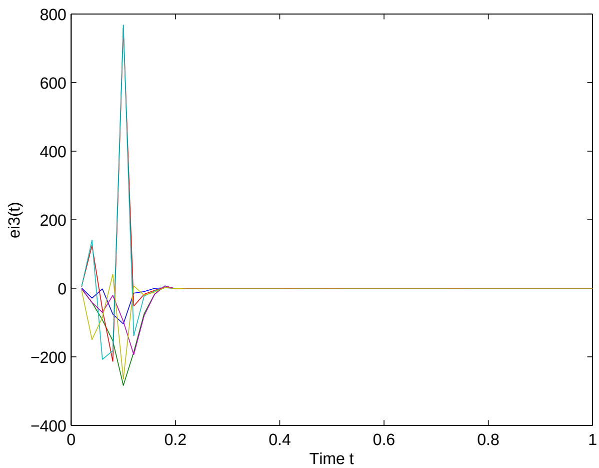

Figure 4. The lag synchronization error $e_{i3}, (1 \leqslant i \leqslant 6)$ between the driver and response networks.

synchronization between the drive and response systems are obtained in Fig. 5, Fig. 6, and Fig. 7.

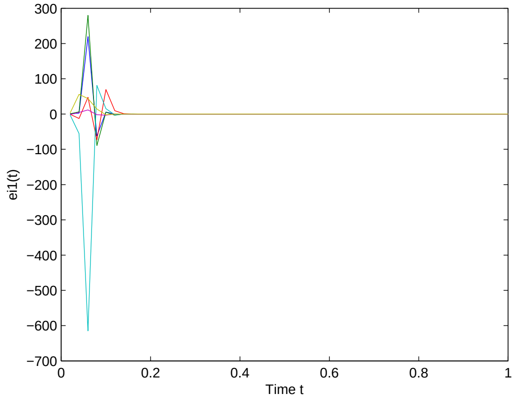

Figure 5. The lag synchronization error $e_{i1}, (1 \leqslant i \leqslant 6)$ between the driver and response networks.

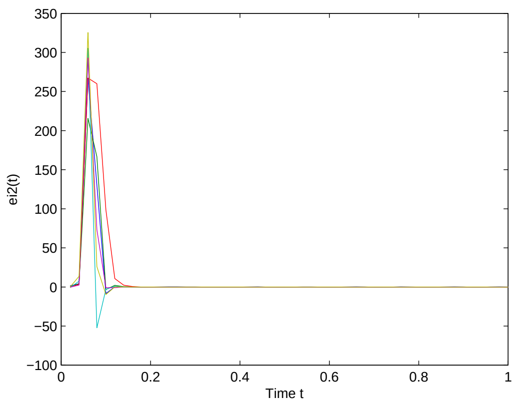

Figure 6. The lag synchronization error $e_{i2}, (1 \leqslant i \leqslant 6)$ between the driver and response networks.

5. Conclusion

This paper investigates the problem of pinning synchronization between two complex dynamical networks with lag. Pinning controllers are designed and some sufficient criteria are obtained through the lemma of Schur complement and Lyapunov stability theorem. Furthermore, the number of controllers in the response system can be adjusted by changing the values of parameters in the systems, and the minimum number of controllers can be easily obtained. At last, numerical examples are provided to verify the correctness of the theoretical results.

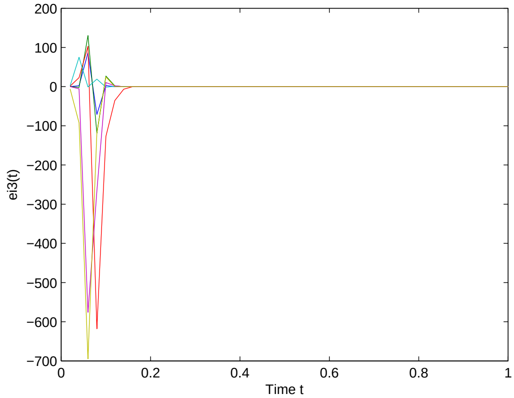

Figure 7. The lag synchronization error $e_{i3}, (1 \leqslant i \leqslant 6)$ between the driver and response networks.

References

[1] D.J. WATTS and S.H. STROGATZ: Collective dynamics of 'small-word' networks. *Nature*, **393** (1998), 440-442.

[2] M. GIRVAN and M.E.J. NEWMAN: Community structure in social and biological networks. *Proc. of the National Academy of Sciences USA*, **99** (2002), 7821-7826.

[3] R. ALBERT, H. JEONG and A.L. BARABÁSI: Diameter of the world-wide web. *Nature*, **401** (1999), 130-131.

[4] R.J. WILLIAMS and N.D. MARTINEZ: Simple rules yield complex food webs. *Nature*, **404** (2000), 180-183.

[5] P. ERDÖS and A. RÉNYI: On the evolution of random graphs. *Publications of the Mathematical Institute of the Hungarian Academy of Sciences*, **5** (1960), 17-61.

[6] E. ALMAAS, R.V. KULKARNI and D. STOUD: Characterizing the structure of small-world networks. *Physical Review Letters*, **88** (2002), 098101.1-4.

[7] A.L. BARABÁSI and R. ALBERT: Emergence of scaling in random networks. *Science*, **286** (1999), 509-512.

[8] A.L. BARABÁSI, R. ALBERT and H. JEONG: Mean-field theory for scale-free random networks. *J. of Physics A*, **272** (1999), 173-187.

[9] Y. FAN, Y. WANG, Y. ZHANG and Q. WANG: Robust synchronization control for complex networks with disturbed sampling couplings. *Applied Mathematics and Computation*, **219** (2013), 6719Ű6728.

[10] F. NIAN, X. WANG, Y. NIU and D. LIN: Module-phase synchronization in complex dynamic system. *Applied Mathematics and Computation*, **217** (2010), 2481-2489.

[11] D. GHOSH: Projective-dual synchronization in delay dynamical systems with time-varying coupling delay. *Nonlinear Dynamics*, **66** (2011), 717-730.

[12] P. RAO, Z. WU and M. LIU: Adaptive projective synchronization of dynamical networks with distributed time delays. *Nonlinear Dynamics*, **67** (2012), 1729-1736.

[13] H. DU, P. SHI and N. LU: Function projective synchronization in complex networks with time delay via hybrid feedback control. *Nonlinear Analysis: Real World Applications*, **14** (2013), 1182-1190.

[14] S. ZHENG, G. DONG and Q. BI: Impulsive synchronization of complex networks with non-delayed and delayed coupling. *Physics Letters A*, **373** (2009), 4255-4259.

[15] K. LI and C. LAI: Adaptive impulsive synchronization of uncertain complex dynamical networks. *Physics Letters A*, **372** (2008), 1601-1606.

[16] J. LU, D.W.C. HO, J. CAO and J. KURTHS: Single impulse controller for global exponential synchronization of dynamical networks. *Nonlinear Analysis: Real World Applications*, **14** (2013), 581-593.

[17] K. WANG, X. FU and K. LI: Cluster synchronization in community networks with nonidentical nodes. *Chaos*, **19** (2009), 023106.1-023106.10.

[18] E. GUIREY, M. BEES, A. MARTIN and M. SROKOSZ: Persistence of cluster synchronization under the influence of advection. *Physics Review E*, **81** (2010), 051902.1-059102.16.

[19] W. LU, B. LIU and T. CHEN: Cluster synchronization in networks of coupled nonidentical dynamical systems. *Chaos*, **20** (2010), 013120.1-013120.12.

[20] J. ZHOU, J.LU and J. LÜ: Pinning adaptive synchronization of a general complex dynamical network. *Automatica*, **44** (2008) 996-1003.

[21] C. FAN, G. JIANG and F. JIANG: Synchronization between two complex dynamical networks using scalar signals under pinning control. *IEEE Trans. on Circuits Systems I*, **57** (2011) 2991-2998.

[22] Y.Z. SUN. W. LI and J. RUAN: Generalized outer synchronization between complex dynamical networks with time delay and noise perturbation. *Communications in Nonlinear Science and Numerical Simulation*, **18** (2013), 989-998.

[23] Q. MIAO, Y. TANG, S. LU and J. FANG: Lag synchronization of a class of chaotic systems with unkown parameters. *Nonlinear Dynamics*, **57** (2009), 107-112.

[24] G. HU and Z. QU: Controlling spatiotemporal chaos in coupled map lattice systems. *Physical Review Letters*, **72** (1994), 68-71.

[25] X. JIN and G. YANG: Adaptive pinning synchronization of a class of nonlinearly coupled complex networks. *Communications in Nonlinear Science and Numerical Simulation*, **18** (2013), 316-326.

[26] W. GUO: Lag synchronization of complex networks via pin ning control. *Nonlinear Analysis: Real World Applications*, **12** (2011), 2579-2585.

[27] W. YU and J. CAO: Adaptive synchronization and lag synchronization of uncertain dynamical system with time delay based on parameter identification. *Physics A*, **375** (2007), 467-482.

[28] S. BOYED, L.E. GHAOUI, E. FERON and V. BALAKRISHNAN: Linear matrix inequalities in system and control theory. Philadelphia, PA:SIAM, 1994.

[29] J. ZHOU, J. LU and J. LÜ: Erratum to:"Pinning adaptive synchronization of a general complex dynamical network". [*Automatica*, **44** (2008), 996-1003], *Automatica*, **45** (2009), 598-599.

[30] J. LU, D.W.C. HO: Stability of complex dynamical networks with noise disturbance under performance constant. *Nonlinear Analysis: Real World Applications*, **12** (2011), 1974-1984.

[31] W. LU and T. CHEN: Synchronization of coupled connected neural networks with delays. *IEEE Trans. on Circuits and Systems*, **51** (2004), 2491-2503.