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# Lyapunov matrices approach to the parametric optimization of a system with two delays and a PD-controller

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In the paper the parametric optimization problem for a linear system with two delays and a PD-controller is presented. In the parametric optimization problem the quadratic performance index is considered. The value of the quadratic index of quality is calculated due to the Lyapunov functional and is equal to the value of that functional for the initial function of the neutral system with two delays. The Lyapunov functional is determined by means of the Lyapunov matrix.

**Key words:** neutral system, Lyapunov matrix, Lyapunov functional

## 1. Introduction

The Lyapunov functionals for time delay systems have many applications. For example they are used to test the stability of time delay systems, in calculation of the robustness bounds for uncertain time delay systems, in computation of the exponential estimates for the solutions of time delay systems. The Lyapunov quadratic functionals are also used in the parametric optimization problem of time delay systems to calculation of the value of the quadratic performance index. The value of the Lyapunov functional at the initial function of time delay system is equal to the value of the quadratic index of quality. For the first time, the method of determination of the Lyapunov functional for a time delay system with one delay was presented by Repin [12]. Duda developed the Repin's method of determination of the Lyapunov functional for a system with a time-varying delay [1], for a system with both lumped and distributed delay [3]. Apart from the Repin's method, the method of determination of the Lyapunov functional by means of Lyapunov matrices is very popular, see for example [2, 4–6, 8–11, 13, 14]. Duda used this method in parametric optimization problem for

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a system with one retarded type time delay and a P-controller [2], for a neutral system with one delay and a P-controller [4], for a system with two delays and a P-controller [5] and for a neutral system with two delays and a P-controller [6]. In the paper the parametric optimization problem for a linear system with two delays and a PD-controller is presented. The value of the quadratic index of quality is equal to the value of the Lyapunov functional for the initial function of the neutral system with two delays. The Lyapunov functional is determined by means of the Lyapunov matrix. This paper extends the results of Duda [5, 6] to a system with two delays and a PD-controller. To the best of the author's knowledge, such extension has not been reported in the literature. An example, illustrating the method, is also presented. The paper is organized as follows. In Section 2 is introduced the mathematical model of neutral systems with two delays, is given the formula of the Lyapunov functional for that system and theorem concerning the Lyapunov matrix properties. In Section 3 is given procedure of determination of the Lyapunov matrix for a neutral system with two commensurate delays. The parametric optimization problem is formulated in Section 4. In Section 5 is given an example of parametric optimization problem for a system with two commensurate delays and a PD controller. Conclusions end the paper.

## 2. Preliminaries

Let us consider a neutral system with two delays

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} - D\dfrac{\mathrm{d}x(t-h)}{\mathrm{d}t} = Ax(t) + Bx(t-h) + Cx(t-r), \\ x(\theta) = \varphi(\theta) \end{cases} \tag{1}$$

for $t \geqslant 0$, $\theta \in [-r, 0]$.

The instant state $x(t) \in \mathbb{R}^n$, matrices $A$, $B$, $C$, $D \in \mathbb{R}^{n\times n}$, initial function $\varphi \in PC^1([-r,0],\mathbb{R}^n)$ – the space of piece-wise continuously differentiable vector valued functions defined on the segment $[-r,0]$ with the uniform norm $\|\varphi\|_{PC^1} = \sup\limits_{\theta\in[-r,0]} \|\varphi(\theta)\|$, delays $r > h > 0$.

The theorems of existence, continuous dependence and uniqueness of solutions of Eq. (1) are given in [7].

We assume that the difference $x(t) - Dx(t-h)$ is continuous and differentiable for $t \geqslant 0$, except possibly a countable number of points, $t_j = jh$, $j = 0, 1, 2, \ldots$ and that the matrix $D$ is not singular and Schur stable i.e. its eigenvalues lie in the interior of the unit disk of the complex plane.

In parametric optimization problem will be used the performance index of quality

$$J = \int_0^\infty x^T(t,\varphi)Wx(t,\varphi)\mathrm{d}t, \tag{2}$$

where $W \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and $x(t,\varphi)$ is a solution of Eq. (1) for initial function $\varphi$.

The value of the performance index is equal to the value of the Lyapunov functional $v$ for initial function $\varphi$ [6]

$$\begin{aligned}
v(\varphi) = \int_0^\infty x^T(t,\varphi)Wx(t,\varphi)\mathrm{d}t = \\
= \varphi^T(0)[U(0)-U(-h)D-D^TU^T(-h)+D^TU(0)D]\varphi(0)+ \\
+2\varphi^T(0)\int_{-h}^0 [U(-\theta-h)-D^TU(-\theta)]\left[B\varphi(\theta)+D\frac{\mathrm{d}}{\mathrm{d}\theta}\varphi(\theta)\right]\mathrm{d}\theta+ \\
+2\varphi^T(0)\int_{-r}^0 [U(-\theta-r)-D^TU(h-r-\theta)]C\varphi(\theta)\mathrm{d}\theta+ \\
+\int_{-h}^0\int_{-h}^0 \left[B\varphi(\theta)+D\frac{\mathrm{d}}{\mathrm{d}\theta}\varphi(\theta)\right]^T U(\theta-\xi)\left[B\varphi(\xi)+D\frac{\mathrm{d}}{\mathrm{d}\xi}\varphi(\xi)\right]\mathrm{d}\theta\mathrm{d}\xi+ \\
+2\int_{-h}^0\int_{-r}^0 \left[B\varphi(\theta)+D\frac{\mathrm{d}}{\mathrm{d}\theta}\varphi(\theta)\right]^T U(\theta-\xi+h-r)C\varphi(\xi)\mathrm{d}\theta\mathrm{d}\xi+ \\
+\int_{-r}^0\int_{-r}^0 \varphi^T(\theta)C^TU(\theta-\xi)C\varphi(\xi)\mathrm{d}\theta\mathrm{d}\xi,
\end{aligned} \tag{3}$$

where $U$ is a Lyapunov matrix.

**Theorem 1** [6] *If system (1) is exponentially stable, then the Lyapunov matrix for that system fulfills the following conditions:*

- *the dynamic property*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}U(\xi)-\frac{\mathrm{d}}{\mathrm{d}\xi}U(\xi-h)D = U(\xi)A+U(\xi-h)B+U(\xi-r)C \tag{4}$$

*for $\xi \geqslant 0$ and $\xi \neq jh$, $j = 0,1,2,\ldots$;*

- *the symmetry property*
$$U(-\xi) = U^T(\xi) \tag{5}$$
*for* $\xi \geqslant 0$;
- *the algebraic property*
$$\begin{aligned} -W = {} & A^T U(0) + U(0)A - A^T U(-h)D - D^T U^T(-h)A + \\ & + B^T U^T(-h) + U(-h)B - B^T U(0)D - D^T U(0)B + \\ & + C^T U^T(-r) + U(-r)C - C^T U(r-h)D - D^T U(h-r)C. \end{aligned} \tag{6}$$

## 3. A Lyapunov matrix for a neutral system with two commensurate delays

Let us consider a neutral system with two commensurate delays

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} - D\dfrac{\mathrm{d}x(t-h)}{\mathrm{d}t} = Ax(t) + Bx(t-h) + Cx(t-2h), \\ x(\theta) = \varphi(\theta) \end{cases} \tag{7}$$

for $t \geqslant 0$, $\theta \in [-2h, 0]$.

The instant state $x(t) \in \mathbb{R}^n$, matrices $A$, $B$, $C$, $D \in \mathbb{R}^{n\times n}$, initial function $\varphi \in PC^1([-2h,0],\mathbb{R}^n)$ - the space of piece-wise continuously differentiable vector valued functions defined on the segment $[-2h,0]$ with the uniform norm $\|\varphi\|_{PC^1} = \sup_{\theta\in[-2h,0]} \|\varphi(\theta)\|$, delays $h$, $2h > 0$.

A set of Eqs. (4), (5), (6) for a system (7) takes a form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau) - \frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau-h)D = U(\tau)A + U(\tau-h)B + U(\tau-2h)C, \tag{8}$$

$$U(-\tau) = U^T(\tau) \tag{9}$$

for $\tau \in [0, 2h]$

$$\begin{aligned} -W = {} & A^T U(0) + U(0)A - A^T U(-h)D - D^T U^T(-h)A + \\ & + B^T U^T(-h) + U(-h)B - B^T U(0)D - D^T U(0)B + \\ & + C^T U^T(-2h) + U(-2h)C - C^T U(h)D - D^T U(-h)C. \end{aligned} \tag{10}$$

Formula (9) extends the function $U$ defined on the segment $[0, 2h]$ to the segment $[-2h, 0]$.

Indeed for $\tau \in [0, 2h]$, $U(-\tau) = U^T(\tau)$. For $\varsigma = -\tau$, $U(\varsigma) = U^T(-\varsigma)$ and $\varsigma \in [-2h, 0]$.

We split the function $U$, defined on the interval $[-2h, 2h]$, into four partial functions and we shift each partial function to the segment $[0, h]$. These partial functions, defined for $\xi \in [0, h]$, are denoted as $U_1(\xi), U_2(\xi), Z_1(\xi), Z_2(\xi)$, and are defined by formulas

$$U_1(\xi) = U(\xi), \tag{11}$$

$$U_2(\xi) = U(h+\xi), \tag{12}$$

$$Z_1(\xi) = U(\xi - h) = U^T(-\xi + h), \tag{13}$$

$$Z_2(\xi) = U(\xi - 2h) = U^T(-\xi + 2h). \tag{14}$$

Relations (11)–(14) imply

$$U(0) = U_1(0), \quad U(-h) = Z_1(0), \quad U(-2h) = Z_2(0), \quad U(h) = U_2(0). \tag{15}$$

Taking into account (15) the algebraic property (10) can be written in a form

$$\begin{aligned} -W = {} & A^T U_1(0) + U_1(0)A - A^T Z_1(0)D - D^T Z_1^T(0)A + \\ & + B^T Z_1^T(0) + Z_1(0)B - B^T U_1(0)D - D^T U_1(0)B + C^T Z_2^T(0) + \\ & + Z_2(0)C - C^T U_2(0)D - D^T Z_1(0)C. \end{aligned} \tag{16}$$

We will use the relations

$$U(-\xi) = U^T(\xi) = U_1^T(\xi), \tag{17}$$

$$U(-\xi - h) = U^T(\xi + h) = U_2^T(\xi), \tag{18}$$

$$U(2h - \xi) = U^T(\xi - 2h) = Z_2^T(\xi) \tag{19}$$

for $\xi \in [0, h]$.

Taking into account relations (11)–(14), Eq. (8) can be written in a form:

- for $\tau = \xi$, $d\tau = d\xi$, where $\xi \in [0, h]$

$$\frac{\mathrm{d}}{\mathrm{d}\xi} U_1(\xi) - \frac{\mathrm{d}}{\mathrm{d}\xi} Z_1(\xi)D = U_1(\xi)A + Z_1(\xi)B + Z_2(\xi)C; \tag{20}$$

- for $\tau = \xi + h$, $d\tau = d\xi$, where $\xi \in [0, h]$

$$\frac{\mathrm{d}}{\mathrm{d}\xi} U_2(\xi) - \frac{\mathrm{d}}{\mathrm{d}\xi} U_1(\xi)D = U_2(\xi)A + U_1(\xi)B + Z_1(\xi)C; \tag{21}$$

- for $\tau = -\xi + h$, $d\tau = -d\xi$, where $\xi \in [0, h]$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi} U(-\xi + h) - \frac{\mathrm{d}}{\mathrm{d}\xi} U(-\xi)D = {} & -U(-\xi + h)A + \\ & - U(-\xi)B - U(-\xi - h)C; \end{aligned} \tag{22}$$

- for $\tau = -\xi + 2h$, $d\tau = -d\xi$, where $\xi \in [0,h]$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}U(-\xi+2h) - \frac{\mathrm{d}}{\mathrm{d}\xi}U(-\xi+h)D = -U(-\xi+2h)A+ \\ -U(-\xi+h)B - U(-\xi)C. \qquad (23)$$

We transpose both sides of Eq. (22) and taking into account relations (11)–(14), (17) and (18) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\xi}Z_1(\xi) - D^T\frac{\mathrm{d}}{\mathrm{d}\xi}U_1(\xi) = -A^T Z_1(\xi) - B^T U_1(\xi) - C^T U_2(\xi). \qquad (24)$$

We transpose both sides of Eq. (23) and taking into account relations (11)–(14), (17) and (19) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\xi}Z_2(\xi) - D^T\frac{\mathrm{d}}{\mathrm{d}\xi}Z_1(\xi) = -A^T Z_2(\xi) - B^T Z_1(\xi) - C^T U_1(\xi). \qquad (25)$$

Eqs. (20) and (24) can be reshape to a form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}U_1(\xi) - D^T\frac{\mathrm{d}}{\mathrm{d}\xi}U_1(\xi)D = -A^T Z_1(\xi)D - B^T U_1(\xi)D - C^T U_2(\xi)D+ \\ +U_1(\xi)A + Z_1(\xi)B + Z_2(\xi)C, \qquad (26)$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}Z_1(\xi) - D^T\frac{\mathrm{d}}{\mathrm{d}\xi}Z_1(\xi)D = D^T U_1(\xi)A + D^T Z_1(\xi)B+ \\ +D^T Z_2(\xi)C - A^T Z_1(\xi) - B^T U_1(\xi) - C^T U_2(\xi). \qquad (27)$$

We have obtained a set of ordinary differential equations with unknown $U_1(\xi), U_2(\xi), Z_1(\xi), Z_2(\xi)$.

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\xi}U_1(\xi) - D^T\frac{\mathrm{d}}{\mathrm{d}\xi}U_1(\xi)D = -A^T Z_1(\xi)D - B^T U_1(\xi)D+ \\ \qquad -C^T U_2(\xi)D + U_1(\xi)A + Z_1(\xi)B + Z_2(\xi)C, \\ \frac{\mathrm{d}}{\mathrm{d}\xi}U_2(\xi) - \frac{\mathrm{d}}{\mathrm{d}\xi}U_1(\xi)D = U_2(\xi)A + U_1(\xi)B + Z_1(\xi)C, \\ \frac{\mathrm{d}}{\mathrm{d}\xi}Z_1(\xi) - D^T\frac{\mathrm{d}}{\mathrm{d}\xi}Z_1(\xi)D = D^T U_1(\xi)A + D^T Z_1(\xi)B+ \\ \qquad +D^T Z_2(\xi)C - A^T Z_1(\xi) - B^T U_1(\xi) - C^T U_2(\xi), \\ \frac{\mathrm{d}}{\mathrm{d}\xi}Z_2(\xi) - D^T\frac{\mathrm{d}}{\mathrm{d}\xi}Z_1(\xi) = -A^T Z_2(\xi) - B^T Z_1(\xi) - C^T U_1(\xi), \end{cases} \qquad (28)$$

for $\xi \in [0,h]$ with initial conditions

$$U_1(0),\ \ U_2(0),\ \ Z_1(0),\ \ Z_2(0).$$

Eq. (28) can be written in a form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{bmatrix} col\,U_1(\xi) \\ col\,U_2(\xi) \\ col\,Z_1(\xi) \\ col\,Z_2(\xi) \end{bmatrix} = \mathscr{H}\begin{bmatrix} col\,U_1(\xi) \\ col\,U_2(\xi) \\ col\,Z_1(\xi) \\ col\,Z_2(\xi) \end{bmatrix}, \tag{29}$$

for $\xi \in [0,h]$ with initial conditions

$$col\,U_1(0),\;\; col\,U_2(0),\;\; col\,Z_1(0),\;\; col\,Z_2(0).$$

Solution of the set of ordinary differential equations (29) is given in a form

$$\begin{bmatrix} col\,U_1(\xi) \\ col\,U_2(\xi) \\ col\,Z_1(\xi) \\ col\,Z_2(\xi) \end{bmatrix} = \Phi(\xi)\begin{bmatrix} col\,U_1(0) \\ col\,U_2(0) \\ col\,Z_1(0) \\ col\,Z_2(0) \end{bmatrix}, \tag{30}$$

where a matrix

$$\Phi(\xi) = \begin{bmatrix} \Phi_{11}(\xi) & \Phi_{12}(\xi) & \Phi_{13}(\xi) & \Phi_{14}(\xi) \\ \Phi_{21}(\xi) & \Phi_{22}(\xi) & \Phi_{23}(\xi) & \Phi_{24}(\xi) \\ \Phi_{31}(\xi) & \Phi_{32}(\xi) & \Phi_{33}(\xi) & \Phi_{34}(\xi) \\ \Phi_{41}(\xi) & \Phi_{42}(\xi) & \Phi_{43}(\xi) & \Phi_{44}(\xi) \end{bmatrix} \tag{31}$$

is a fundamental matrix of system (29).

We determine the initial conditions $col\,U_1(0)$, $col\,U_2(0)$, $col\,Z_1(0)$, $col\,Z_2(0)$. Relations (11)-(14) imply

$$U_1(h) = U(h) = U_2(0), \tag{32}$$

$$Z_1(h) = U(0) = U_1(0), \tag{33}$$

$$Z_2(h) = U(-h) = Z_1(0). \tag{34}$$

Solution of the differential equations (29) for $\xi = h$ is given

$$\begin{aligned} col\,U_1(h) = col\,U_2(0) = \Phi_{11}(h)col\,U_1(0) + \Phi_{12}(h)col\,U_2(0) + \\ + \Phi_{13}(h)col\,Z_1(0) + \Phi_{14}(h)col\,Z_2(0), \end{aligned} \tag{35}$$

$$\begin{aligned} col\,Z_1(h) = col\,U_1(0) = \Phi_{31}(h)col\,U_1(0) + \Phi_{32}(h)col\,U_2(0) + \\ + \Phi_{33}(h)col\,Z_1(0) + \Phi_{34}(h)col\,Z_2(0), \end{aligned} \tag{36}$$

$$\begin{aligned} col\,Z_2(h) = col\,Z_1(0) = \Phi_{41}(h)col\,U_1(0) + \Phi_{42}(h)col\,U_2(0) + \\ + \Phi_{43}(h)col\,Z_1(0) + \Phi_{44}(h)col\,Z_2(0). \end{aligned} \tag{37}$$

Eqns. (35) to (37) and (16) enables us to calculate the initial conditions of system (29). We reshape them to a form

$$\Phi_{11}(h) col\, U_1(0) + (\Phi_{12}(h) - 1) col\, U_2(0) + \Phi_{13}(h) col\, Z_1(0) + \\ + \Phi_{14}(h) col\, Z_2(0) = 0, \tag{38}$$

$$(\Phi_{31}(h) - 1) col\, U_1(0) + \Phi_{32}(h) col\, U_2(0) + \Phi_{33}(h) col\, Z_1(0) + \\ + \Phi_{34}(h) col\, Z_2(0) = 0, \tag{39}$$

$$\Phi_{41}(h) col\, U_1(0) + \Phi_{42}(h) col\, U_2(0) + (\Phi_{43}(h) - 1) col\, Z_1(0) + \\ + \Phi_{44}(h) col\, Z_2(0) = 0, \tag{40}$$

$$A^T U_1(0) + U_1(0)A - A^T Z_1(0)D - D^T Z_1^T(0)A + B^T Z_1^T(0) + Z_1(0)B + \\ - B^T U_1(0)D - D^T U_1(0)B + C^T Z_2^T(0) + Z_2(0)C - C^T U_2(0)D + \\ - D^T Z_1(0)C = -W. \tag{41}$$

## 4. Formulation of the parametric optimization problem

Let us consider a system with two delays and a PD-controller

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + B_1 u(t-h) + Cx(t-r), \\ u(t) = -Px(t) - T_d \dfrac{\mathrm{d}x(t)}{\mathrm{d}t}, \\ x(\theta) = \varphi(\theta), \end{cases} \tag{42}$$

for $t \geqslant 0$, $\theta \in [-r, 0]$, where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $A$, $C \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times p}$, $P, T_d \in \mathbb{R}^{p\times n}$, $\varphi \in PC^1([-r,0], \mathbb{R}^n)$.

System (42) can be written in an equivalent form

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} + B_1 T_d \dfrac{\mathrm{d}x(t-h)}{\mathrm{d}t} = Ax(t) - B_1 Px(t-h) + Cx(t-r), \\ x(\theta) = \varphi(\theta). \end{cases} \tag{43}$$

In parametric optimization problem will be used the performance index

$$J = \int_0^\infty x^T(t,\varphi) W x(t,\varphi) \mathrm{d}t, \tag{44}$$

where $W \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and $x(t,\varphi)$ is a solution of Eq. (43) for initial function $\varphi$.

**Problem 1.** We are looking for the matrices $P, T_d \in \mathbb{R}^{p\times n}$ which minimize the integral quadratic performance index (44.

The value of the performance index (44) is equal to the value of the functional (3) for initial function $\varphi$, in which matrices $D$ and $B$ should be replaced by matrices $-B_1T_d$ and $-B_1P$ respectively. To calculate the value of the functional (3) we need a Lyapunov matrix $U(\xi)$.

The optimization problem is formulated for neutral system with two arbitrary time delays but it will be solved for neutral system with two commensurate delays because there is not method of solving of the equations in Theorem 1 for any two arbitrary non commensurate delays.

## 5. Example

Let us consider a system with a PD-controller

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = ax(t) + b_1u(t-h) + cx(t-2h), \\ u(t) = -px(t) - T_d\dfrac{dx(t)}{\mathrm{d}t}, \\ x(\theta) = \varphi(\theta), \end{cases} \tag{45}$$

$t \geqslant 0$, $x(t)$, $u(t) \in \mathbb{R}$, $\theta \in [-2h, 0]$, $h \geqslant 0$, $a$, $b_1$, $c$ – real numbers, $p$, $T_d$ – PD-controller parameters, $\varphi$ is an initial function of a system.

One can reshape Eq. (45) to a form

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} + b_1T_d\dfrac{\mathrm{d}x(t-h)}{\mathrm{d}t} = ax(t) - b_1px(t-h) + cx(t-2h), \\ x(\theta) = \varphi(\theta), \end{cases} \tag{46}$$

for $t \geqslant 0$ and $\theta \in [-2h, 0]$.

In parametric optimization problem we use the performance index of quality

$$J = \int_0^\infty wx^2(t,\varphi)\mathrm{d}t, \tag{47}$$

where $w > 0$ and $x(t,\varphi)$ is a solution of Eq. (46) for initial function $\varphi$.

The Lyapunov functional for a system (46) has a form, see formula (3)

$$v(\varphi)=\left[(1+b_1^2T_d^2)U(0)+2b_1T_dU(-h)\right]\varphi^2(0)+$$
$$-2\varphi(0)b_1\int_{-h}^{0}\left[U(-\theta-h)+b_1T_dU(-\theta)\right]\left[p\varphi(\theta)+T_d\frac{\mathrm{d}\varphi(\theta)}{\mathrm{d}\theta}\right]\mathrm{d}\theta+$$
$$+2\varphi(0)c\int_{-2h}^{0}\left[U(-\theta-2h)+b_1T_dU(-h-\theta)\right]\varphi(\theta)\mathrm{d}\theta+$$
$$+b_1^2\int_{-h}^{0}\int_{-h}^{0}U(\theta-\eta)\left[p\varphi(\theta)+T_d\frac{\mathrm{d}\varphi(\theta)}{\mathrm{d}\theta}\right]\left[p\varphi(\eta)+T_d\frac{\mathrm{d}\varphi(\eta)}{\mathrm{d}\eta}\right]\mathrm{d}\eta\mathrm{d}\theta+$$
$$-2b_1c\int_{-h}^{0}\int_{-2h}^{0}U(-h+\theta-\eta)\left[p\varphi(\theta)+T_d\frac{\mathrm{d}\varphi(\theta)}{\mathrm{d}\theta}\right]\varphi(\eta)\mathrm{d}\theta\mathrm{d}\eta+$$
$$+c^2\int_{-2h}^{0}\int_{-2h}^{0}U(\theta-\eta)\varphi(\theta)\varphi(\eta)\mathrm{d}\theta\mathrm{d}\eta. \tag{48}$$

The value of the performance index of quality (47) is equal to the value of the functional (48) for initial function $\varphi$

$$J=v(\varphi). \tag{49}$$

To obtain the value of the performance index of quality one needs a Lyapunov matrix $U(\xi)$ for $\xi\in[0,2h]$. In Section 3 was presented a method of determination of the Lyapunov matrix for a system with two delays.

System of Eqs. (28) takes a form

$$\begin{bmatrix}\frac{\mathrm{d}}{\mathrm{d}\xi}U_1(\xi)\\ \frac{\mathrm{d}}{\mathrm{d}\xi}U_2(\xi)\\ \frac{\mathrm{d}}{\mathrm{d}\xi}Z_1(\xi)\\ \frac{\mathrm{d}}{\mathrm{d}\xi}Z_2(\xi)\end{bmatrix}=\begin{bmatrix}h_1 & h_2 & h_3 & h_7\\ h_4 & h_5 & h_6 & -h_2\\ -h_3 & -h_7 & -h_1 & -h_2\\ -h_6 & h_2 & -h_4 & -h_5\end{bmatrix}\begin{bmatrix}U_1(\xi)\\ U_2(\xi)\\ Z_1(\xi)\\ Z_2(\xi)\end{bmatrix}, \tag{50}$$

where

$$h_1=\frac{a-b_1^2pT_d}{1-b_1^2T_d^2},\qquad h_2=\frac{b_1cT_d}{1-b_1^2T_d^2},\qquad h_3=\frac{ab_1T_d-b_1p}{1-b_1^2T_d^2},$$

$$h_4 = \frac{-ab_1T_d - b_1p + 2b_1^3pT_d}{1-b_1^2T_d^2}, \qquad h_5 = \frac{a - ab_1^2T_d^2 - b_1^2cT_d^2}{1-b_1^2T_d^2},$$

$$h_6 = \frac{c - b_1^2cT_d^2 + b_1^2pT_d - ab_1^2T_d^2}{1-b_1^2T_d^2}, \qquad h_7 = \frac{c}{1-b_1^2T_d^2}.$$

Initial conditions of system (50) one obtains solving the algebraic equation

$$\begin{bmatrix} \Phi_{11}(h) & \Phi_{12}(h)-1 & \Phi_{13}(h) & \Phi_{14}(h) \\ \Phi_{31}(h)-1 & \Phi_{32}(h) & \Phi_{33}(h) & \Phi_{34}(h) \\ \Phi_{41}(h) & \Phi_{42}(h) & \Phi_{43}(h)-1 & \Phi_{44}(h) \\ 2(a-b_1^2pT_d) & cb_1T_d & 2(ab_1T_d-b_1p)+cb_1T_d & 2c \end{bmatrix} \cdot \begin{bmatrix} U_1(0) \\ U_2(0) \\ Z_1(0) \\ Z_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -w \end{bmatrix}, \tag{51}$$

where $\Phi(\xi)$ is a fundamental matrix of solutions of Eq. (50).

We search for optimal PD controller parameters $p_{\text{opt}}$ and $T_{d_{\text{opt}}}$ which minimize the index (47) for the initial function $\varphi$ given by a formula

$$\varphi(\theta) = \begin{cases} x_0 & \text{for } \theta = 0, \\ 0 & \text{for } \theta \in [-2h, 0). \end{cases} \tag{52}$$

For the initial function $\varphi$ given by the formula (52) the performance index of quality has a form

$$J = v(\varphi) = U_1(0)x_0^2. \tag{53}$$

Figure 1 shows the value of the index $J(p)$ for fixed $T_d = 0.8957$ and $h = 1$. You can see that there exists a critical value of the gain $p_{crit}$. The system (46) is stable for gains less than critical one and unstable for gains grater than critical.

Figure 2 shows the value of the index $J(p)$ for fixed $T_d = 0.8957$, $h = 1$ and gains less the critical one. You can see that the function $J(p)$ is convex and has a minimum.

Figure 3 shows the value of the index $J(T_d)$ for fixed $p = 0.3358$ and $h = 1$. There exists a critical value of the differential time $T_{d\,crit}$ too, which determines the interval of stability.

Figure 4 shows the value of the index $J(T_d)$ for fixed $p = 0.3358$, $h = 1$ and $T_d$ less the critical one. You can see that the function $J(T_d)$ is convex and has a minimum.

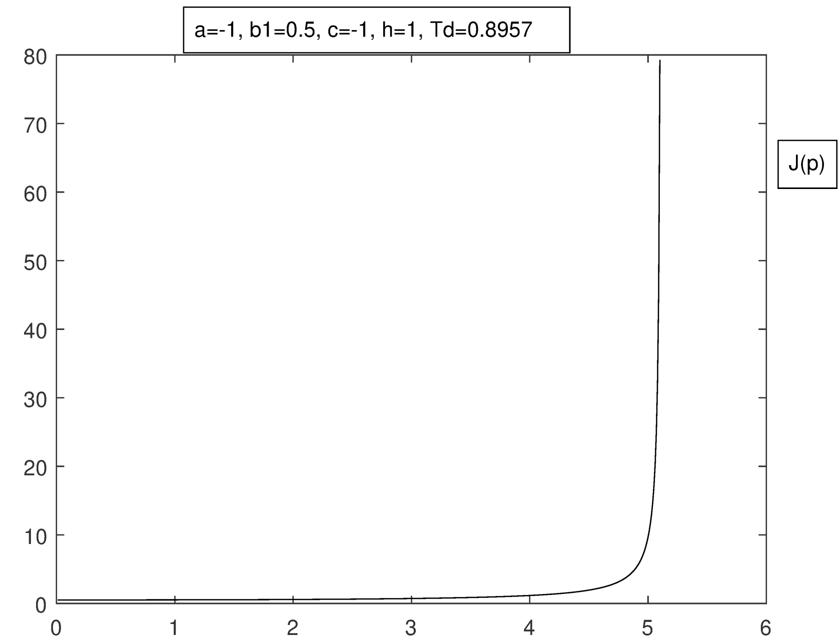


Figure 1: Value of the index $J(p)$ for fixed $T_d = 0.8957$ and $h = 1$

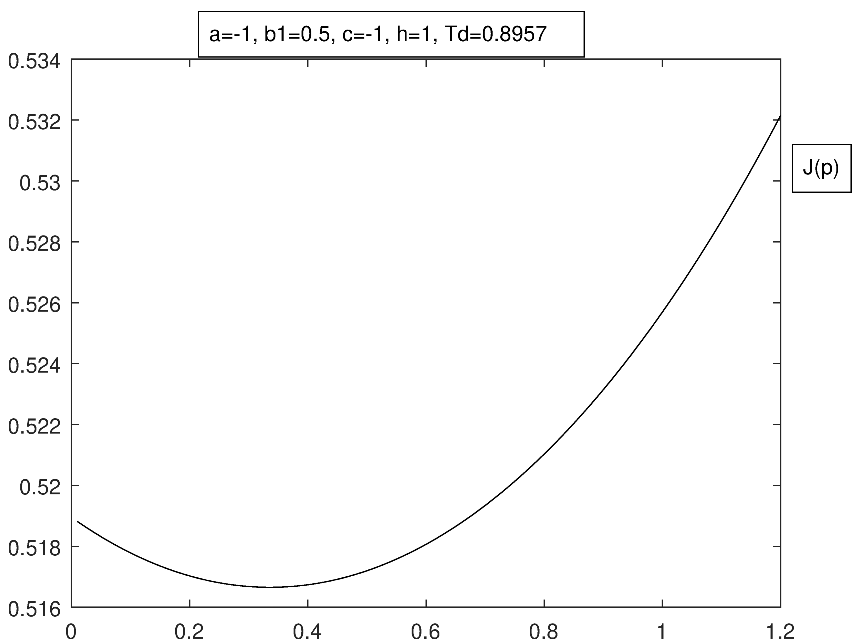


Figure 2: Value of the index $J(p)$ for fixed $T_d = 0.8957$ and $h = 1$

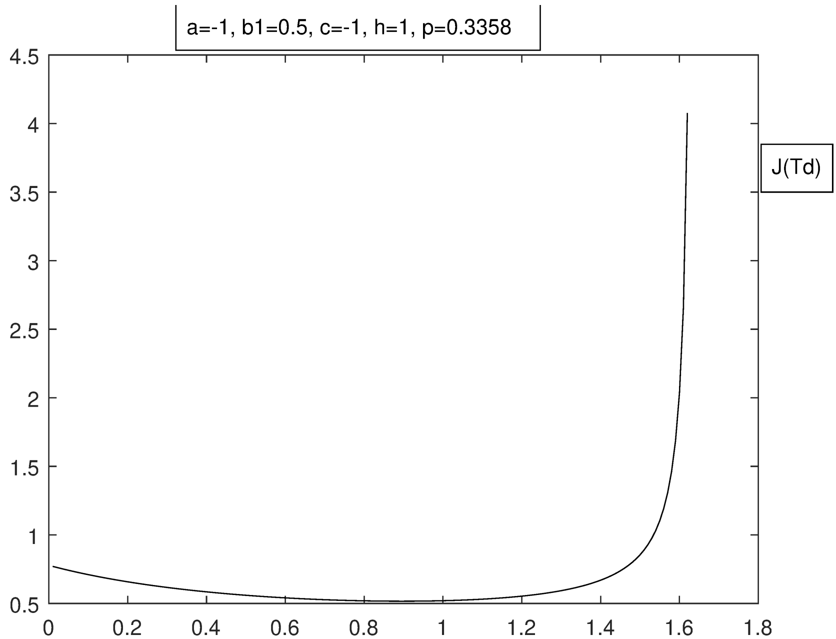


Figure 3: Value of the index $J(T_d)$ for fixed $p = 0.3358$ and $h = 1$

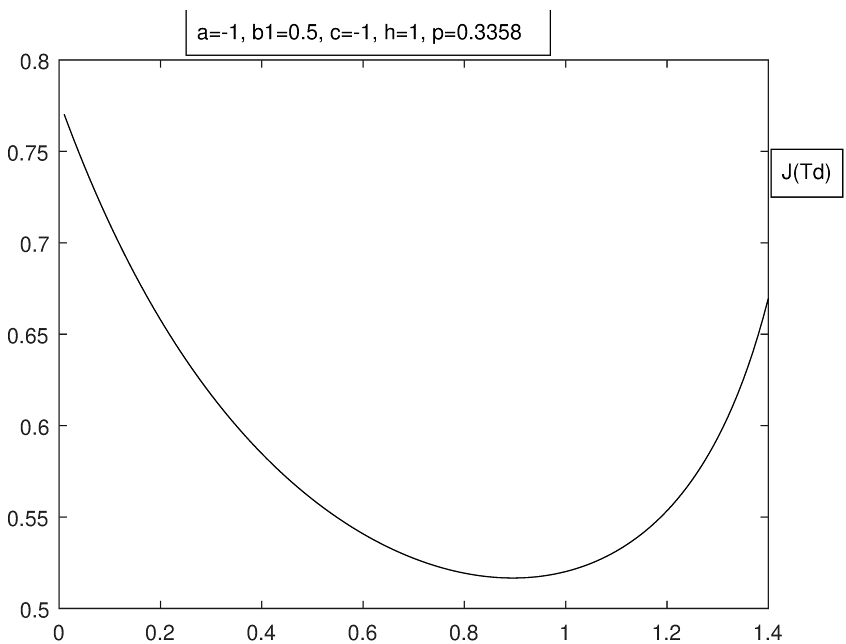


Figure 4: Value of the index $J(T_d)$ for fixed $p = 0.3358$ and $h = 1$

We search for an optimal parameters of a PD-controller which minimize the index (53). Optimization results, obtained by means of Matlab function *fminsearch*, are given in Table 1. These results are obtained for $x_0 = 1$, $w = 1$, $a = -1$, $b_1 = 0.5$, $c = -1$.

Table 1: Optimization results for $a = -1$, $b_1 = 0.5$, $c = -1$

| $h$ | $p_{\text{opt}}$ | $T_{d\,\text{opt}}$ | $J(p, T_d)$ |
|---|---|---|---|
| 0.25 | 4.8567 | 0.9619 | 0.2031 |
| 0.50 | 1.7038 | 0.9575 | 0.3323 |
| 0.75 | 0.7348 | 0.9350 | 0.4313 |
| 1.00 | 0.3358 | 0.8957 | 0.5167 |
| 1.25 | 0.1594 | 0.8490 | 0.5953 |
| 1.50 | 0.0804 | 0.8028 | 0.6702 |
| 1.75 | 0.0481 | 0.7606 | 0.7427 |
| 2.00 | 0.0397 | 0.7233 | 0.8134 |

Critical values $p_{crit}$ and $T_{d\,\text{crit}}$ depend on the value of time delay. This dependence is presented in Table 2. Critical gain is obtained for fixed $T_d = 0.8957$ and critical differential time is obtained for fixed $p = 0.3358$.

Table 2: Critical gain and differential time

| delay $h$ | $p_{\text{crit}}$ | $T_{d\,\text{crit}}$ |
|---|---|---|
| 0.25 | 16.0 | 1.89 |
| 0.50 | 8.7 | 1.80 |
| 0.75 | 6.3 | 1.71 |
| 1.00 | 5.1 | 1.62 |
| 1.25 | 4.3 | 1.54 |
| 1.50 | 3.9 | 1.50 |
| 1.75 | 3.5 | 1.42 |
| 2.00 | 3.3 | 1.37 |

Figure 5 shows graphs of functions $U_1(\xi)$, $U_2(\xi)$, $Z_1(\xi)$ and $Z_2(\xi)$ obtained with the Matlab code, for parameters of system (46) used in optimization process with $h = 1$ and for optimal values of the PD controller parameters $p_{opt} = 0.3358$ and $T_{d_{opt}} = 0.8957$.

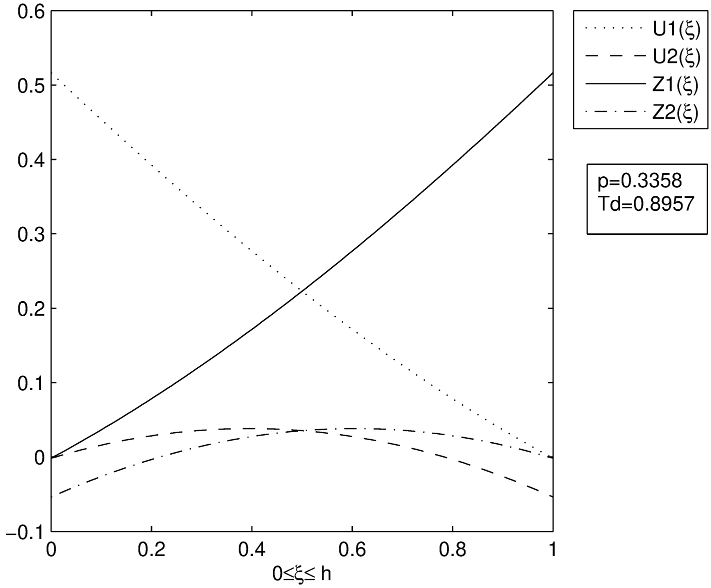


Figure 5: Graphs of functions $U_1(\xi)$, $U_2(\xi)$, $Z_1(\xi)$, $Z_2(\xi)$

## 6. Conclusions

In the paper the Lyapunov matrices approach to the parametric optimization problem of a system with two delays and a PD controller is presented. Using of the PD controller in the time delay system of retarded type changes the system to the time delay system of neutral type. The value of the integral quadratic performance index is equal to the value of the Lyapunov functional for the initial function of a neutral system. The Lyapunov functional is determined by means of the Lyapunov matrix.

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